PMM U.S.S.R.,Vol.45,pp.223-230 0021-8928/82/2 0223 $7.50/0

UDC 539.3

ANTIPLANE PROBLEM FOR A SYSTEM OF LINEAR INCLUSIONS IN AN ISOTROPIC MEDIUM*

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A method is proposed for the solution of the problem of antiplane shear of two half-spaces with a thin-walled tape inclusion of arbitrary mechanical nature at the interface of materials. The results are extended to the case of systems of curved inclusions in a homogeneous space and half-space. Underlying the method is modeling the inclusions by a previously unknown jump in the tangential stresses and displacements. The dependence of the state of stress of the medium on functions of the jump and the external loading is determined. Two generalized stress intensity coefficients are introduced into the considerations, and asymptotic expressions are established for the stress and displacement fields in the neighborhood of the edge of the inclusion. Interaction conditions between the elastic inclusion and the medium are constructed, and an example of an elastic inclusion on the interface of materials is examined. The influence of the shape of the inclusion, which is variable from the elliptic to the rectangular according to a definite dependence, on the stress concentration.

The problem of antiplance deformation of a body with an elastic arc inclusion was examined earlier in /1/. The theme of this paper adjoins investigations of antiplane problems for bodies with slits /2/, and because of the known analogy is close to questions of stationary heat conductivity for bodies with heat permeable cracks /3/.

1. Inclusion on the interface separating the materials. Let a thin-walled inclusion be on a straight line separating two materials that coincides with the Ox axis of the xOy coordinate syatem. The complex variable $z = x + iy$ and the coordinate z must be differentiated. The external load is determined by the homogeneous tangential stress field at infinity $\tau_{yz}^\infty = \tau$, $\tau_{xz}^\infty = \tau_k$, as well as by the forces Q_k acting at the points $z_{ck} \in S_k$ ($k = 2$ for the upper half-plane S_2, and $k = 1$ for the lower half plane S_1). The inclusion is assumed so thin that its action can be modeled by a certain jump in the stress τ_{yz} and the displacements w on the middle line L_0' of the interlayer:

$$\tau_{yz}^- - \tau_{yz}^+ = f_5(x),\ w^{-\prime} - w^{+\prime} = f_6(x)\quad (x \in L_0) \tag{1.1}$$

Here and henceforth, the superscripts plus and minus denote the limit values of the appropriate quantities on the upper and lower edges of the line L_0 representing the abscissa axis; $w = w(x, y)$ is the displacement in the Oz direction, the prime denotes the derivative with respect to the argument, and $f_5(x) = f_6(x) = 0$ for $x \overline{\in} L_0'$.

Taking account of the Hooke's law

$$\tau_{yz} = \mu \partial w / \partial y,\quad \tau_{xz} = \mu\, \partial w / \partial x \tag{1.2}$$

where μ is the shear modulus of the material at the point under consideration, we give the relationships (1.1) the form

$$\tau_{yz}^- - \tau_{yz}^+ = f_5(x),\quad \tau_{xz}^- / \mu_1 - \tau_{xz}^+ / \mu_2 = f_6(x)\quad (x \in L_0) \tag{1.3}$$

The representation

$$\tau_{yz} + i\tau_{xz} = \omega'(z),\quad w = [\operatorname{Im} \omega(z)] / \mu_k\ (z \in S_k) \tag{1.4}$$
$$\omega'(z) = \tau + i\tau_k + iS(z) + \omega_0'(z),\quad S(z) = \sum_{k=1,2} \{Q_k [2\pi (z - z_{ck})]^{-1}\}$$

holds for the stress components and the displacement w in each of the half-planes, where $\omega_0'(z)$ is a holomorphic function in each of the half-planes separately, and tends to zero at infinity.

Substituting the first of the relationships (1.4) into the conditions (1.3) and solving

*Prikl.Matem.Mekhan.,45,No.2,308-318,1981 the corresponding problem of the linear conjugate of the limit values just as this is done in /4/, we obtain

$$\omega'(z) = ip_k t_5^{\circ}(z) - ct_6^{\circ}(z) + c_k(z), \quad c_k(z) = \tau + i[\tau_k + S(z) + n_{kl}p_k S^*(z)] \tag{1.5}$$

$$t_r^m(z) = \frac{1}{\pi}\int\limits_{L_m'} \frac{f_r(t)\,dt}{t-z}, \quad S^*(z) = S_l(z) + \overline{S_k}(z)$$

$$p_k = \frac{\mu_k}{\mu_1+\mu_2}, S_k(z) = Q_k[2\pi(z - z_{ck})]^{-1}$$

$$n_{kl} = \frac{\mu_k - \mu_l}{\mu_k}, \quad c = p_k\mu_l \quad (z \in S_k;\ r = 5, 6;\ k, l = 1, 2;\ k \neq l)$$

as well as the condition $\tau_1\mu_2 = \tau_2\mu_1$ for the stress at infinity.

Therefore

$$\tau_{yz}(z) + i\tau_{xz}(z) = ip_k t_5^{\circ}(z) - ct_6^{\circ}(z) + \tau_{yz}^{\circ}(z) + i\tau_{xz}^{\circ}(z) \tag{1.6}$$

where $\tau_{yz}^{\circ}(z) + i\tau_{xz}^{\circ}(z) = c_k(z)$ is the stress at the appropriate point of the domain in the absence of inclusions. Application of the Sokhotskii—Plemelj formula permits determination of the limit values of the stresses

$$\tau_{yz}^{\pm}(z) = \mp p_k f_5(x) - ct_6^{\circ}(x) + \tau_{yz}^{\circ\pm}(x), \quad \tau_{xz}^{\pm}(x) = \mp cf_6(x) + p_k t_5^{\circ}(x) + \tau_{xz}^{\circ\pm}(x) \tag{1.7}$$

The value $k = 2$ corresponds to the upper sign, and $k = 1$ to the lower.

2. System of rectilinear inclusions in an isotropic body. Let us consider $N + 1$ thin-walled inclusions in a homogeneous isotropic plane (space) (Fig.1). The external load is determined by the stresses at infinity $\tau_{yz0}^{\infty} = \tau$, $\tau_{xz0}^{\infty} = \tau_1$, and the force Q_1 at the point z_{c1}° of the main coordinate system $x_0O_0y_0$. The middle line L_p' of the p-th inclusion ($p = 0, 1, \ldots, N$) belongs to the abscissa axis L_p of the local system $x_pO_py_p$ defining the complex variable $z_p = x_p + iy_p$. The coordinates of the point O_p in the main system are defined by the values $z_0 = z_{0p}$, and the axis x_p makes the angle α_p with the x_0 axis ($z_{00} = 0, \alpha_0 = 0$).

In the absence of inclusions, the external load determines the potential

$$\omega_{00}'(z_0) = \tau + i\tau_1 + iS_0^{\circ}(z_0)$$
$$S_0^p(z_p) = Q_1[2\pi(z - z_{c1}^p)]^{-1}, \quad z_{c1}^p = (z_{c1}^{\circ} - z_{0n})\exp(-i\alpha_p)$$

in the main system.

For one inclusion along the line L_m' of the homogeneous plane, which determines the stress jump $f_{5m}^*(x) = f_{5m}(x) + i\mu f_{6m}(x)$ according to (1.5), the complex potential in the m-th coordinate system is

$$\omega_m'(z_m) = \frac{i}{2}t_{5m}^{m*}(z_m), \quad t_{5m}^{m*}(z_m) = t_{5m}^m(z_m) + i\mu t_{6m}^m(z_m)$$

Denoting the expressions for the potentials $\omega_m'(z_m)$, $\omega_{00}'(z_0)$ in the p-th system by $\omega_m^{p'}(z_p)$, $\omega_{0p}'(z_p)$, the stresses in the m-th system by τ_{xzm}, τ_{yzm}, and taking account of the equality $\tau_{yz0} + i\tau_{xz0} = (\tau_{yzm} + i\tau_{xzm})\exp(-i\alpha_m)$, we can write /1/

$$\omega_m^{p'}(z_p) = \omega_m'(z_m^p)\exp(i\alpha_{pm}), \quad \omega_{0p}'(z_p) = iS_0^p(z_p) + (\tau + i\tau_1)\exp(i\alpha_p)$$
$$z_m^p = (Z_p - z_{0m})\exp(-i\alpha_m), \quad Z_p = z_p\exp(i\alpha_p) + z_{0p}, \quad \alpha_{pm} = \alpha_p - \alpha_m$$

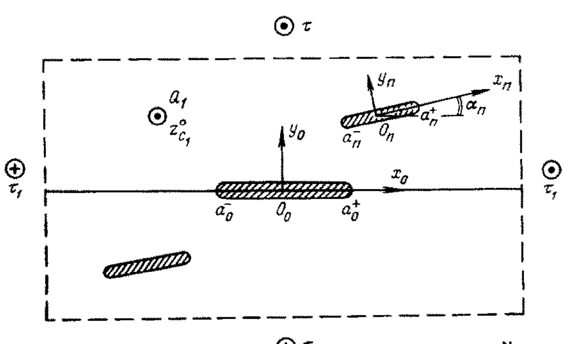

Fig.1

On the basis of the superposition principle, which is completely applicable under these conditions, the solution of the problem posed is determined, in the p-th coordinate system, by the potential

$$\omega^{p'}(z_p) = \sum_{m=0}^{N}\omega_m^{p'}(z_p) + \omega_{0p}(z_p) \tag{2.1}$$

Then the stresses at an arbitrary point of the plane are

$$\tau_{yzp}(z_p) + i\tau_{xzp}(z_p) = \omega^{p'}(z_p) = \tag{2.2}$$

$$\frac{i}{2\pi}\sum_{m=0}^{N}\int\limits_{L_m'} P_{pm}(t, z)f_{5m}^*(t)\,dt + \mathring{\tau}_{yzp}(z_p) + i\tau_{xzp}(z_p) \quad (p = 0, 1, \ldots, N)$$

where $\overset{\circ}{\tau}_{yzp}, \overset{\circ}{\tau}_{xzp}$ are stress tensor components at an appropriate point in the plane in the absence of inclusions

$$\overset{\circ}{\tau}_{yzp}(z_p) + i\overset{\circ}{\tau}_{xzp}(z_p) = iS_0^p(z_p) + (\tau + i\tau_1)\exp(i\alpha_p) \tag{2.3}$$
$$P_{pm}(t, z) = (T_m - Z_p)^{-1}\exp(i\alpha_p), \quad T_m = t\exp(i\alpha_m) + z_{0m}$$

Utilization of the Sokhotskii—Plemelj formula yields the following expressions for the limit values on the line L_p':

$$\tau_{yzp}^{\pm}(x_p) + i\tau_{xzp}^{\pm}(x_p) = \mp\frac{i}{2}f_{5p}^*(x_p) + \frac{i}{2\pi}\sum_{m=0}^{N}\int_{L_m'} P_{pm}(t, x)f_{5m}^*(t)\,dt + \overset{\circ}{\tau}_{yzp}(x_p) + i\overset{\circ}{\tau}_{xzp}(x_p) \tag{2.4}$$

$$P_{pm}(t, x) = (T_m - X_p)^{-1}\exp(i\alpha_p), \quad X_p = x_p\exp(i\alpha_p) + z_{0p}, \quad (p = 0, 1, \ldots, N)$$

Formulas (2.2) and (2.4) hold for any functions of the jump. If they are unknown, conditions for the interaction between the inclusion and the matrix

$$\varphi_p(\tau_{xzp}^{\pm}, \tau_{yzp}^{\pm}) = 0 \quad (p = 0, 1, \ldots, N) \tag{2.5}$$

that follow from an examination of some mechanical model of the inclusions, must be used. Substituting (2.4) into the conditions (2.5) results in a system of integral equations in the jump function. Moreover, these functions should satisfy additional conditions that follow from physical considerations, of the form, for instance

$$\int_{L_p'} f_{5p}^*(t)\,dt = A_p = \mathrm{const} \quad (p = 0, 1, \ldots, N) \tag{2.6}$$

As an example, if the first p_1 inclusions are modeled by slits on whose edges the stresses $\tau_q^{\pm}(x_q)$ $(q = 0, 1, \ldots, p_1 - 1)$ are given, while the rest are represented by stiff inclusions inserted with the tension $w_r^{\pm}(x_r)$ $(r = p_1, \ldots, N)$, then such interaction conditions have the form

$$\tau_{yzq}^{\pm}(x_q) = \tau_q^{\pm}(x_p), \quad \tau_{xzr}^{\pm}(x_r) = \mu w_r^{\pm\prime}(x_r) \quad (x_q \in L_q', \quad x_r \in L_r') \tag{2.7}$$

and the additional conditions have the form (2.6) for $A_p = 0$.

In particular, for $p_1 = N + 1$, $\tau_q^+ = \tau_q^-$, there follows $f_{5q}(x_q) = 0$ and a system of integral equations for a set of longitudinal shear cracks /2/ from (2.7) and (2.4). If the inclusions are elastic, then relationships of the form (6.5) and (6.8) can be used. Application of other interaction conditions permits a study of distinct models of elastic inclusions /1/ or interlayers with other mechanical properties: plastic, liquid, etc.

3. A system of inclusions in a half-space. Let $L_0' = [-\infty, \infty]$, and let the other inclusions be in the lower half-plane. Then on the basis of the inversion formula for a singular integral equation /5/, it follows from the relationship (2.4) taken for the value $p = 0$, that

$$f_{50}^*(\eta) = \frac{2i}{\pi}\int_{-\infty}^{\infty}\left[\tau_{yz0}^{\pm} + i\tau_{xz0}^{\pm} \pm \frac{1}{2}f_{50}^* - \overset{\circ}{\tau}_{yz0} - i\overset{\circ}{\tau}_{xz0}\right]\frac{dt}{t-\eta} + \frac{i}{\pi}\sum_{m=1}^{N}\int_{L_m'} P_{0m}(t, \eta)f_{5m}^*(t)\,dt \tag{3.1}$$

Defining $f_{50}(\eta)$ as the real part of (3.1), we have after having substituted the expression obtained into the remaining relationship (2.4) for $p = 0, 1, \ldots, N$:

$$\tau_{yzp}^{\pm}(x_p) + i\tau_{xzp}^{\pm}(x_p) = \mp{}^1/_2 f_{5p}^*(x_p) + I^{\pm}(x_p) - I_x(x_p) + \overset{\circ}{\tau}_{yzp}(x_p) + i\overset{\circ}{\tau}_{xzp}(x_p) \tag{3.2}$$

$$I^{\pm}(x_p) = \frac{i}{2\pi}\sum_{m=1}^{N}\int_{L_m}[P_{pm}^{\mp}(t, x)f_{5m}(t) + \mu iP_{pm}^{\pm}(t, x)f_{6m}(t)]\,dt$$

$$I_a(x_p) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_{p0}(t, x)[\tau_{az0}^{-}(t) - \overset{\circ}{\tau}_{az0}(t)]\,dt$$

$$(a = x, y, p = 1, 2, \ldots, N)$$

$$P_{pm}^{\pm}(t, x) = P_{pm}(t, x) \pm P_{pm}^*(t, x), \quad P_{pm}^*(t, x) = (\overline{T}_m - X_p)^{-1}\exp(i\alpha_p)$$

Analogously defining the function $f_{60}(\eta)$ as the imaginary part of $f_{50}^*(\eta)$ from (3.1), we obtain

$$\tau_{yzp}^{\pm}(x_p)+i\tau_{xzp}^{\pm}(x_p)=\mp{}^1/_2 f_{5p}^*(x_p)+I^-(x_p)+\ iI_y(x_p)\ +\overset{\bullet}{\tau}_{yzp}(x_p)+i\overset{\bullet}{\tau}_{xzp}(x_p)\quad (p=1,2,\ldots,N) \tag{3.3}$$

The following values of the integrals are used in deriving (3.1) and (3.2):

$$\int_{-\infty}^{\infty}\{P_{0m}(t,\eta),P_{p0}(\eta,t)\}\frac{d\eta}{\eta-x}=\pi i\{P_{0m}(t,x),P_{p0}(x,t)\} \tag{3.4}$$
$$\int_{-\infty}^{\infty}\{P_{0m}(t,\eta),\overline{P_{0m}(t,\eta)}\}P_{p0}(\eta,x)\,d\eta=\{0,-2\pi iP_{pm}^*(t,x)\}$$

It is convenient to use (3.2) for displacements given on the half-space boundary, when $w^{-\prime}(x)\equiv\tau_{xz0}^-(x)/\mu$, and (3.3) is taken when the tangential stresses τ_{yz0}^- are known on the boundary. If x_p is replaced formally by z_p in (3.2) and (3.3), and the terms $\mp{}^1/_2 f_{5p}^*(x_p)$ are discarded, then we obtain an expression for the stress at an arbitrary point of the half-plane.

4. Inclusions of curvilinear configuration. The results obtained above are generalized naturally to the case when L_p' are arbitrary smooth curves given parametrically by the equation $z_0=z_p^*(x)\equiv x_p^*(x)+iy_p^*(x)$. Because of the possibility of considering such a line as the limit of a set of contiguous rectilinear segments, we note that the relationships (2.2) are conserved if only we set

$$T_p=z_p^*(x),\quad \alpha_p\sim\alpha_p(t)\equiv\operatorname{arctg}[y_p^{*\prime}(t)/x_p^{*\prime}(t)] \tag{4.1}$$
$$f_{5p}^*(t)=\tau_{nzp}^-(t)-\tau_{nzp}^+(t)+i[\tau_{\tau zp}^-(t)-\tau_{\tau zp}^+(t)]$$

Here the n,τ among the subscripts govern the direction of the normal and the tangent of the line L_p' at the appropriate point $z_p^*(t)$. In this case (2.4) is converted into

$$\tau_{nzp}^{\pm}(x)+i\tau_{\tau zp}^{\pm}(x)=\mp\frac{1}{2}f_{5p}^*(x)+\frac{i}{2\pi}\sum_{m=0}^{N}\int_{L_m'}P_{pm}(t,x)\times f_{5m}^*(t)\,dt+\overset{\circ}{\tau}_{nzp}(x)+i\overset{\circ}{\tau}_{\tau zp}(x)\quad (p=0,1,\ldots,N) \tag{4.2}$$

where $X_p=z_p^*(x),\alpha_p\sim\alpha_p(x)$, and x is the value of the natural parameter on the arc. The expressions (3.2) and (3.3) are modified similarly.

The interaction conditions for the inclusion and the matrices for curvilinear inclusions should be represented, analogously to (2.4), in the form $\Psi_p(\tau_{nzp}^{\pm},\ \tau_{\tau zp}^{\pm})=0\ (p=0,1,\ldots,N)$. From (4.2) and the conditions $\tau_{nzp}^{\pm}=0\ (p=0,1,\ldots,N)$ and taking account of the analogy in /3/, a system of singular integral equations for the antiplane problem follows for a system of curvilinear slots /6/. Utilization of simplified interaction conditions (conditions (11) in /3/) permits obtaining the result of /1/ for an elastic arclike inclusion.

5. Asymptotic expressions. If functions of the jump at the endfaces of the p-th inclusion have a root singularity (utilization of the conditions (2.7) or (6.5), (6.8) results in precisely such a case), then on the basis of the formula /5/

$$t_{5p}^{p*}(z)=\mp p_{5p}^{\pm}r^{-1/2}\exp(-i\theta/2)+O(1),\quad p_{5p}^{\pm}=\lim_{s_p\to 0}[\sqrt{s_p}\,f_{5p}^*(x)] \tag{5.1}$$

as well as the relationships (2.1), (2.2), (1.4), the asymptotic expressions

$$\begin{Bmatrix}\tau_{yzp}\\ \tau_{xzp}\end{Bmatrix}=\frac{k_3^{1p}}{\sqrt{2r}}\begin{Bmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\end{Bmatrix}+\frac{k_3^{2p}}{\sqrt{2r}}\begin{Bmatrix}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{Bmatrix}+O(1) \tag{5.2}$$

$$\begin{Bmatrix}\tau_{rzp}\\ \tau_{\theta zp}\end{Bmatrix}=\frac{k_3^{1p}}{\sqrt{2r}}\begin{Bmatrix}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{Bmatrix}+\frac{k_3^{2p}}{\sqrt{2r}}\begin{Bmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\end{Bmatrix}+O(1) \tag{5.3}$$

$$w=\frac{\sqrt{2r}}{\mu}\left[k_3^{1p}\sin\frac{\theta}{2}+k_3^{2p}\cos\frac{\theta}{2}\right]+O(r^{3/2}) \tag{5.4}$$

follow, where

$$k_3^{2p}-ik_3^{1p}=\mp p_{5p}^{\pm}/\sqrt{2} \tag{5.5}$$

τ_{rzp}, $\tau_{\theta zp}$ are tangential stress components in a local polar coordinate system at the end of the p-th inclusion, s_p is the distance between the point $z_p^*(x)$ on the line L_p' and its end $a_p^\pm$, and the upper sign is selected in (5.1) and (5.5) if the right endface of this inclusion is considered, and the lower sign, if the left endface is considered, k_3^{2p}, k_3^{1p} are generalized stress intensity coefficients for the antiplane strain (the stiff and flexible parts of the stress distribution in the terminology of /7/). The expressions (5.2) and (5.4) agree with those obtained in /7/ for tapered elastic inclusions of arbitrary shape, particularly, hypocycloidal and hypotrochoidal.

Formulas (5.3) and (5.4) also hold for inclusions of smooth curvilinear shape. To use (5.2) in this case it is necessary to orient the x_p axis of the local coordinate system along the tangent direction at the appropriate end of the line L_p'.

6. Interaction conditions for an elastic inclusion and matrix. Let us consider an inclusion of small thickness $2h(x)$ whose axis of symmetry is the segment $L_0' = [a_0^-, a_0^+]$. Along $L_0^{k'}$ the inclusion makes contact with the domains S_k which, if necessary, can be represented in the form of half-planes with shear modulus μ_k, respectively, $(k = 1, 2)$ (Fig.2).

Using the equilibrium condition $\partial\tau_{yzb}/\partial y + \partial\tau_{xzb}/\partial x = 0$ and integrating first with respect to y between the limits $-h$ and h, and then with respect to x between a_0^- and x, we obtain

$$\tau_{xz}^c(x) \equiv \frac{1}{2h(x)}\int_{-h}^{h}\tau_{xzb}dy = \tau_{xz}^c(a_0^-) - \frac{1}{2h(x)}\int_{a_0^-}^{x}(\tau_{yzb}^2 - \tau_{yzb}^1)\,dt \tag{6.1}$$

where the subscript b denotes a quantity referring to the inclusion, while the superscript $k = 1, 2$ refers to the value of the functions on the line $L_0^{k'}$. On the other hand, taking account of the slight thickness of the inclusion and of (1.2), we have

$$\tau_{xz}^c(x) \approx {}^1/_2(\tau_{xzb}^2 + \tau_{xzb}^1) = {}^1/_2\mu_b\partial/\partial x \times (w_b^2 + w_b^1) \tag{6.2}$$

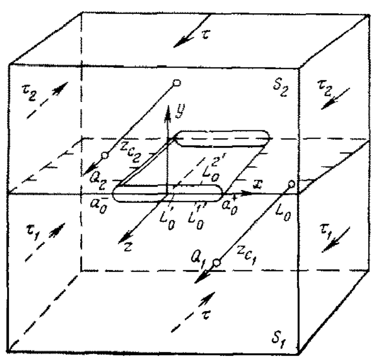

Fig.2

The conditions for ideal mechanical contact between the inclusion and the matrices can be represented in the form

$$w = w_b,\quad \tau_{yz} \equiv \mu_k\frac{\partial w}{\partial y} = \tau_{yzb} \equiv \mu_b\frac{\partial w_b}{\partial y}\quad \text{on } L_0^{k'},\quad (k = 1, 2) \tag{6.3}$$

Here and henceforth in this section w is the displacement field in the matrix, and w_0 is the corresponding displacement in the absence of an inclusion which satisfies the conditon

$$w_0^+ = w_0^-,\quad \mu_2\partial w_0^+/\partial y = \mu_1\partial w_0^-/\partial y \tag{6.4}$$

Taking into account the approximate equality $w(x \pm ih) \approx w^\pm(x) \pm h\partial w_0(x \pm ih)/\partial y$ as well as the conditions (6.3) and (6.4) and the smallness of the quantity h, we obtain

$$\tau_{yz}^+ + \tau_{yz}^- \approx \tau_{yz}(x + ih) + \tau_{yz}(x - ih) = \mu_b\,[(w^+ - w^-)/h + \tau_{yz}^\circ(x + ih)/\mu_2 + \tau_{yz}^\circ(x - ih)/\mu_1] \tag{6.5}$$

$$w^+ - w^- = \int_{a_0^-}^{x}\left(\frac{\tau_{xz}^+}{\mu_2} - \frac{\tau_{xz}^-}{\mu_1}\right)dt + w\_^*\quad (w\_^* = \text{const}) \tag{6.6}$$

If it is assumed that $w(x \pm ih) \approx w^\pm(x)$, then a more simpler expression will hold

$$\tau_{yz}^+ + \tau_{yz}^- = \mu_b(w^+ - w^-)/h \tag{6.7}$$

On the basis of the contact conditions (6.3) and (6.1) and (6.2), we will have

$$\mu_b(w^{+\prime} + w^{-\prime}) = 2\tau_{xz}^c(a_0^-) - \frac{1}{h(x)}\int_{a_0^-}^{x}(\tau_{yz}^+ - \tau_{yz}^-)\,dt \tag{6.8}$$

The relationships (6.8) and (6.5) or (6.7) comprise the interaction conditions for the inclusion and the matrices. Let us note that upon substituting (1.7), (2.4), (3.2) or (3.3) into these conditions during the construction of the governing system of integral equations for $\tau_{yz}^{\pm}, \tau_{xz}^{\pm}$, it is more expedient to replace $\tau_{yz}^{\circ}(x), \tau_{xz}^{\circ}(x)$ by $\tau_{yz}^{\circ}(x \pm ih), \tau_{xz}^{\circ}(x \pm ih)$.

7. Example. Let us examine the case of a rectilinear elastic inclusion along the segment $L_0' = [-a, a]$, the line separating the materials, in more detail. Substituting (1.7) into conditions (6.5) and (6.8) with the remark made at the end of Sect.6 taken into account, results in the system of equations

$$t_6^c(x) + \beta_6 f_5(x) - \frac{\alpha_6}{h(x)} \int_{-a}^{x} f_6(t)\,dt = F_6(x) \qquad (|x| \leqslant a) \tag{7.1}$$
$$t_5^c(x) + \beta_5 f_6(x) - \frac{\alpha_5}{h(x)} \int_{-a}^{x} f_5(t)\,dt = F_5(x)$$

with the additional conditions

$$\int_{-a}^{a} f_5(t)\,dt = \int_{-a}^{a} f_6(t)\,dt = 0 \tag{7.2}$$

Here

$$\beta_5 = n_{21}\mu_2 / 2, \quad \beta_6 = n_{21} / (2\mu_1), \quad \alpha_5 = \mu_b / (2p_2\mu_1), \quad \alpha_6 = \mu_2 / (2p_2\mu_b) \tag{7.3}$$
$$F_5(x) = \mu_2 [2\tau_{xz}^c(-a) / \mu_b - \tau_{xz}^{\circ}(x + ih) / \mu_2 - \tau_{xz}^{\circ}(x - ih) / \mu_1] / (2p_2)$$
$$F_6(x) = [\mu_{b2}\tau_{yz}^{\circ}(x + ih) + \mu_{b1}\tau_{yz}^{\circ}(x - ih) - \mu_b w_-^+] / (2c), \quad \mu_{bk} = (\mu_k - \mu_b) / \mu_k$$

The quantities w_-^* and $\tau_{xz}^c(-a)$ must be determined from a priori formulas.

As $h \to 0$, and $\mu_b = 0$ or $\mu_b = \infty$, results for a crack and an absolutely stiff film on the material interface follow, respectively:

$$f_5(x) = 0, \quad t_6^{\circ}(x) = \tau_{yz}^{\circ}(x) / c \equiv \Phi_6(x) \quad \text{(necessary is } \tau_{xz}^c(-a) = 0) \tag{7.4}$$
$$f_6(x) = 0, \quad t_5^{\circ}(x) = -\tau_{xz}^{\circ +}(x) / p_2 \equiv \Phi_5(x) \quad \text{(necessary is } w_-^* = 0) \tag{7.5}$$

The solution of the integral equations (7.4) and (7.5) is presented in /5/, and we have in the absence of concentrated forces:

For a crack

$$f_5(x) = 0, \quad f_6(x) = \tau x / [cX(x)], \quad \tau_{yz}^{\pm} = 0, \quad \tau_{xz}^{\pm} = \mp\tau x / X(x), \quad w^{\pm} = \pm\tau X(x) / \mu_k$$
$$k_3^1 = \tau\sqrt{a}, \quad k_3^2 = 0 \qquad (|x| \leqslant a)$$

For an absolutely rigid film

$$f_5(x) = -2\tau_1 x / X(x), \quad f_6(x) = 0, \quad \tau_{yz}^{\pm} = \pm\tau_1 x / X(x), \quad \tau_{xz}^{\pm} = 0,$$
$$w^{\pm} = 0$$
$$k_3^1 = 0, \quad k_3^2 = \tau_1\sqrt{a} \qquad (|x| \leqslant a)$$

For a real absolutely rigid inclusion with nonzero thickness, we should take $\mu_b = \infty, w_-^* = 0$, and then (7.1) is converted into

$$\int_{-a}^{x} f_6(t)\,dt = -h(x)\left[\frac{\overset{\circ}{\tau}_{yz}(x + ih)}{\mu_2} + \frac{\overset{\circ}{\tau}_{yz}(x - ih)}{\mu_1}\right] \qquad (|x| \leqslant a) \tag{7.6}$$
$$t_5^{\circ}(x) + \beta_5 f_6(x) = -\frac{\mu_2}{2p_2}\left[\frac{\overset{\circ}{\tau}_{xz}(x + ih)}{\mu_2} + \frac{\overset{\circ}{\tau}_{xz}(x - ih)}{\mu_1}\right]$$

It can be shown that the displacement jump $w(x + ih) - w(x - ih)$ at the ends of a rigid inclusion is zero within the framework of the approximation in Sect.6, because of (7.6).

If the materials of the half-planes and of the inclusion are identical $(\mu_1 = \mu_2 = \mu_b)$, then (7.1) yields $f_5(x) = f_6(x) = 0$, which results in the solution for a homogeneous plane (space).

The approaches represented by other authors /1,3/ for the solution of the antiplane shear problem for bodies with elastic inclusions do not permit obtaining so accurate a soluition of the problem in the last two cases.

We have $\beta_5 = \beta_6 = 0$ when the mechanical characteristics of the half-plane materials are equal $(\mu_1 = \mu_2 = \mu)$, then (7.1) simplify considerably. Then the solution of (7.1), (7.2) for $h(x) = h_0[1 - (x/a)^2]^{1/(2q)}$ $(h_0 = \text{const}, q \geqslant 1)$ is sought in the form of a series of Chebyshev polynomials of the first kind with the root singularity extracted

$$f_r(x) = \frac{a}{\sqrt{a^2 - x^2}} \sum_{p=1}^{\infty} A_p{}^r T_p(x/a) \quad (r = 5, 6;\ |x| \leqslant a) \tag{7.7}$$

To determine the coefficients $A_p{}^r$ of the expansion with the use of the procedure of the method or orthogonal polynomials /8/, a system of linear algebraic equations is constructed

$$\frac{\pi}{2} A^r_{k+1} + \alpha_{0r} \sum_{p=1} A_p{}^r H^q_{pk} = f_k{}^r \quad (k = 0, 1, \ldots) \tag{7.8}$$

$$H^q_{pk} = \frac{\pi\Gamma(2\sigma)}{p4^\sigma} \sum_{j=0,1} \frac{\sin(p_j\pi)(-1)^j}{\Gamma(1+\sigma+p_j)\Gamma(\sigma-p_j)},$$

$$f_k{}^r = \int_{-1}^{1} F_r(at)\sqrt{1-t^2}\, U_k(t)\, dt$$

$$\alpha_{0r} = \alpha_r / h_0, \quad \sigma = 1 - 1/(2q), \quad 2p_j = k + (-1)^j p$$

Let us note that for inclusions of elliptical $(q = 1)$ and rectangular profile $(q = \infty)$

$$H^1_{pk} = \frac{\pi}{2p}\delta_{p,\,k+1}, \quad H^\infty_{pk} = -\frac{4(k+1)\sin^2(p_0\pi)}{[(k+2)^2 - p^2][k^2 - p^2]}$$

Therefore, for $q = 1$ the solution of the system (7.8) is written explicitly $A_p{}^r = 2pf^r_{p-1} / [\pi(p + \alpha_0)]$. Moreover, if there are not concentrated forces, then $F_r(x) = F_r = \text{const}$ and then:

$$A_p{}^r = F_r\delta_{p,1} / (p + \alpha_0), \quad f_r(x) = aF_r / [(1+\alpha_0)\sqrt{a^2 - x^2}]$$

The generalized stress intensity coefficients can be represented by the sum

$$k_3{}^2 - ik_3{}^1 = \mp\frac{\sqrt{a}}{2} \sum_{p=1}^{\infty} (\pm 1)^p (A_p{}^5 + i\mu A_p{}^6)$$

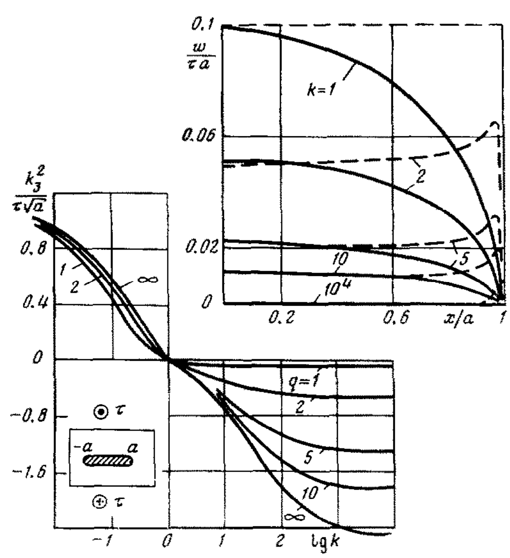

Fig.3

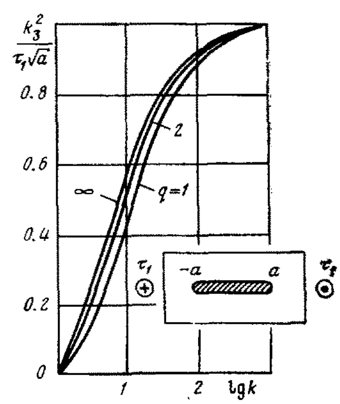

Fig.4

The dependence of the generalized stress intensity coefficients on the parameters k and q under the effect of stresses at infinity has been investigated numerically. It was assumed in the computations

$$k = \mu_b / \mu, \quad a/h_0 = 10, \quad w_-{}^* = 0, \quad \tau_{xz}{}^\circ(-a) = \tau_{xz}{}^\circ(-a) / \max(1, k^{-1})$$

To assure 1% accuracy in the calculations, it turned out to be sufficient to limit ourselves to not more than 25 of the first nonzero coefficients (for the load $A_{2p}{}^r = 0,\ p = 1, 2, \ldots$)

in the expansion (7.7). The results in /9/ afforded a foundation for convergence of the calculation process.

The dependence of $k_3^1 (k_3^2 = 0)$ on the parameters k and q for $\tau_1 = 0$ is represented in Fig.3. Displacement of the upper edge of the inclusion is illustrated by the upper part of Fig.3, where results obtained for $q = 1$ are represented by solid lines, and for $q = 10^4$ by dashes. The calculations verified the tendency of w to approach zero as k increases. It turns out that k_3^1 depends insignificantly on the inclusion shape for relatively pliable inclusions, however, for large k the effect of the shape appears to be sufficiently significant. At the same time, according to /1,3/, the corresponding coefficient is $k_3^1 = 0$ for such external loading and $k = \infty$.

The change in $k_3^2 (k_3^1 = 0)$ is displayed in Fig.4 for $\tau = 0$. This coefficient depends weakly on the shape of the inclusion and is practically zero for $k < 1$. Computations showed that the assumption $\tau_{xz}^c (-a) = 0$ /1/ results, for $k = 1$, in a physically unreal, nonzero value $k_3^2/(\tau/\sqrt{a}) = 0.0909$ for an elliptic, and 0.247 for a rectangular inclusion, which is 9 and 25%, respectively, of its maximum value.

The results presented in this paper are carried over directly /3/ to the corresponding problem of stationary heat conduction of a heat insulated plate with heat conducting inclusions.

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Translated by M.D.F.